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CLOSED-FORM REPRESENTATIONS OF THE LAMBERT W FUNCTION

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Abstract

The Lambert W function is the many-valued analytic inverse of $z(w) = we^w$. We use elementary complex analysis to derive closed-form representations of all of the branches of W through simple quadratures. For instance, if $-\pi < \arg z < \pi$, then the k^{th}, $k = 0, \pm 1, \pm 2, \ldots$, branch of W is given by

$$W_k(z) = \frac{\int\limits_0^\infty \frac{x\,dx}{(x^2+1)B} + \frac{\ln z + 2k\pi\imath}{(\ln z + 2k\pi\imath)^2 + (\pi/2+1)^2}}{\frac{\pi/2+1}{(\ln z + 2k\pi\imath)^2 + (\pi/2+1)^2} - \int\limits_0^\infty \frac{dx}{(x^2+1)B}},$$

where $B = (x - \ln x + \ln z + 2k\pi\imath)^2 + \pi^2$. A similar expression holds true for negative z.

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